

Finite quantum groupoids and inclusions of finite type

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Dedicated to Sergio Doplicher and John E. Roberts on the occasion of their 60th birthday

In the last decade various motivations coming from low dimensional quantum field theory, operator algebra, and Poisson geometry have lead to the introduction of a new notion of symmetry that generalizes both quantum groups and classical groupoid algebras. The slightly different definitions [18, 29, 12, 2, 16, 26, 3, 23, 9] all share in the property that a "quantum groupoid" A contains two antiisomorphic canonical subalgebras: The source subalgebra A^R and the target subalgebra A^L . They reduce to the scalars in the case of a Hopf algebra and they are the carrying space of the trivial representation in the monoidal category M_A of A -modules. In the groupoid interpretation A^L and A^R are non-commutative analogues of the algebra of functions on the space of units. The various definitions of quantum groupoid differ in the size and in commutativity of these subalgebras. The most general among them is Lu's *Hopf algebroid* [16], while the (C^*) -weak Hopf algebra of [2, 3] captures the most general "finite quantum groupoid" (see [22] for a review) which has the extra beauty of being selfdual, just like a finite Abelian group or a finite dimensional Hopf algebra. Yamanouchi's generalized Kac algebra and Hayashi's face algebra are special cases of the latter. In [9], generalizing [26], Enock and Vallin introduced the notion of a Hopf bimodule which corresponds to Lu's bialgebroid in the von Neumann algebraic framework. In [10] Enock constructs an antipode for Hopf bimodules making use of modular theory.

The Doplicher-Roberts duality theorem [8] characterizes the symmetric monoidal Abelian C^* -categories with irreducible monoidal unit as representation categories of (uniquely determined) compact groups. In this way it provides an intrinsic definition of internal symmetry of DHR sectors of quantum field theories in space-time dimension greater than 2. In dimension 2 no analogue result is known. In this respect the significance of C^* -weak Hopf algebras is two-fold. They have representation categories such that

- i) the intrinsic (categorical) dimensions of the objects (in the sense of [15]) are not integers

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- ii) and the monoidal unit is reducible allowing different irreducible "vacuum representations" [4].

Property (i) offers a possibility that the braided C^* -categories found in conformal field theory models are equivalent to representation categories of C^* -weak Hopf algebras, although uniqueness cannot be expected. Even non-braided C^* -categories are included, therefore (ii) suggests that topological soliton sectors can also be described by weak Hopf symmetry.

The universal problem to which the answer is a unique quantum groupoid is not known. But inclusions of (unital C^* -, von Neumann) algebras $N \subset M$ are very close to that. In [24] the regular action of a C^* -weak Hopf algebra A on a von Neumann algebra M has been defined. This is a kind of Galois action which allows $M^A \subset M$ to be any reducible finite index depth 2 inclusion of von Neumann algebras with finite dimensional centers. A Galois correspondence has been established in the case of finite index, finite depth inclusions of II_1 factors by Nikshych and Vainerman [20, 21]. The infinite index, depth 2 case has been treated by Enock and Vallin in [9, 10] for arbitrary von Neumann algebras endowed with a regular operator valued weight.

1 Quantum groupoids

1.1 Bialgebroids. We try to formulate the minimal requirements on an algebraic structure which is to describe "symmetries", hence generalizing the notions of groupings, groupoid duals, Hopf algebras, ... etc. It must be a ring A together with a monoidal structure on the category \mathbf{M}_A of its right modules. If \mathbf{M}_A would be a bimodule category ${}_R\mathbf{M}_R$ for some ring R then a monoidal structure would be given. This motivates the

Definition 1.1 Let $R^{op} \xrightarrow{t} A \xleftarrow{s} R$ be a diagram in the category of rings¹ such that the left and right actions of R defined by $r \cdot a := at(r)$, $a \cdot r := as(r)$ make A into an R - R -bimodule. Equivalently, one requires that the images of s and t commute in A . Then the ring and bimodule A together with a monoidal structure on the category \mathbf{M}_A is called a bialgebroid over R if the forgetful functor $\phi_R: \mathbf{M}_A \rightarrow {}_R\mathbf{M}_R$ is strictly monoidal.

This implicit but natural requirement on ϕ_R is equivalent to a comonoid structure $\langle A, \gamma, \pi \rangle$ in ${}_R\mathbf{M}_R$ which is compatible with the ring structure. More precisely $\gamma: A \rightarrow A \otimes_R A$ and $\pi: A \rightarrow R$ are arrows in ${}_R\mathbf{M}_R$ such that

$$(\gamma \otimes_R \text{id}) \circ \gamma = (\text{id} \otimes_R \gamma) \circ \gamma \quad (1.1)$$

$$\lambda \circ (\pi \otimes_R \text{id}) \circ \gamma = \text{id} = \rho \circ (\text{id} \otimes_R \pi) \circ \gamma \quad (1.2)$$

$$(s(r) \otimes_R 1) \gamma(a) = (1 \otimes_R t(r)) \gamma(a), \quad r \in R, a \in A \quad (1.3)$$

$$\gamma(a) \gamma(b) = \gamma(ab), \quad a, b \in A \quad (1.4)$$

$$\gamma(1) = 1 \otimes_R 1, \quad \pi(1) = 1_R \quad (1.5)$$

$$\pi(t(\pi(a))b) = \pi(ab) = \pi(s(\pi(a))b), \quad a, b \in A. \quad (1.6)$$

These are essentially the same axioms as Lu's in [16] except the different formulation of (1.3, 1.4, 1.6). Unfortunately quite some explanations are needed to elucidate the meaning of Eqns (1.3) and (1.4). The problem is that $A \otimes_R A$ is not a ring. It has a sub-bimodule, however, which is. At first notice that the ring A operates on the bimodule ${}_R A_R$ by left multiplication so it is meaningful to write $(a \otimes_R b)(a' \otimes_R b')$

¹ Rings and their morphisms are always assumed unital

for the result of the tensor product of intertwiners $a \otimes_R b$ acting on the element $a' \otimes_R b'$ of the bimodule $A \otimes_R A$. This convention is used in (1.3) and in the definition

$$\Gamma := \{ x \in A \otimes_R A \mid (s(r) \otimes_R 1)x = (1 \otimes_R t(r))x, \ r \in R \}$$

which is an R - R -bimodule and a ring, too. Now (1.3) and (1.4) just say that $\gamma: A \rightarrow \Gamma$ is a ring homomorphism.

For the equivalence of properties (1)–(6) and monoidality of the functor ϕ_R it will suffice here to recall that ϕ_R is isomorphic to the hom-functor $\text{Hom}(A, -)$ and monoidality of a hom-functor is equivalent to a comonoid structure on the object A . This is how γ and π are constructed.

Definition 1.1 is ready made for a kind of Tannaka-Krein theorem which, in its weakest form, can be formulated as the

Lemma 1.2 *Let \mathcal{C} be an additive monoidal category equivalent to a module category \mathbf{M}_A for some ring A . Let furthermore $F: \mathcal{C} \rightarrow {}_R\mathbf{M}_R$ be a strongly monoidal² functor to the category of bimodules over some ring R . Then A carries a bialgebroid structure over R such that \mathcal{C} and \mathbf{M}_A are equivalent as monoidal categories. The same holds with \mathbf{M}_A replaced by $\mathbf{M}_A^{\text{fgp}}$, the category of finitely generated projective A -modules.*

In order to prepare the discussion of the "finite" bialgebroids let us compute here the endomorphism ring of the monoidal unit of \mathbf{M}_A .

Lemma 1.3 *If A is a bialgebroid over R then the monoidal unit U_A of \mathbf{M}_A is the additive group R together with the A -action $r \triangleleft a = \pi(s(r)a)$. Introducing the notation $Z^s := s(R) \cap \text{Center } A$ and $Z^t := t(R) \cap \text{Center } A$ we have that $t^{-1}(Z^t) = s^{-1}(Z^s)$ is a commutative subring Z of R . The endomorphism ring $\text{End } U$ is isomorphic to Z and consists of multiplications in R with elements of Z .*

Proof The action property of \triangleleft follows from (6). Let $\xi \in \text{End } U$. Then it is also an R - R bimodule endomorphism of R , hence $\xi(r) = rz$, $r \in R$, for some $z \in \text{Center } R$. Hence

$$\begin{aligned} as(z) &= a \cdot z = a_{(1)} \cdot \pi(a_{(2)})z = a_{(1)} \cdot \xi(\pi(a_{(2)})) = a_{(1)} \cdot \xi(1_R \triangleleft a_{(2)}) \\ &= a_{(1)} \cdot (\xi(1_R) \triangleleft a_{(2)}) = a_{(1)} \cdot \pi(s(z)a_{(2)}) = a_{(1)} \cdot \pi(t(z)a_{(2)}) \\ &= s(z)a_{(1)} \cdot \pi(a_{(2)}) = s(z)a \end{aligned}$$

for all $a \in A$, therefore $s(z) \in Z^s$. Similarly, one can prove that $t(z) \in Z^t$. Since s and t are sections of π , they are injective and this proves the Lemma. \square

The above Lemma makes it natural to consider the forgetful functor $\phi_Z: \mathbf{M}_A \rightarrow {}_Z\mathbf{M}_Z$ instead of ϕ_R . It has the advantage that A can be reconstructed from it as $\text{End } \phi_Z$, which is not true for ϕ_R . However, ϕ_Z is a monoidal functor only in the relaxed sense. The natural transformation

$$\mu_{V,W}: \phi_Z(V) \otimes_Z \phi_Z(W) \rightarrow \phi_Z(V \square W)$$

²We use the terminology of MacLane's [17] new edition!

is no longer an isomorphism but (suppressing the ϕ_Z 's in the diagram)

$$\begin{array}{ccccc}
 V \otimes_Z (U \otimes_Z W) & \xrightarrow{V \otimes_Z \lambda_W} & V \otimes_Z W & \xrightarrow{\mu_{V,W}} & V \square W \longrightarrow 0 \\
 \downarrow \wr & \nearrow \rho_V \otimes_Z W & & & \\
 (V \otimes_Z U) \otimes_Z W & & & &
 \end{array} \quad (1.7)$$

is an exact sequence. Together with the inclusion map $\zeta: Z \rightarrow R$ the monoidal functor $\langle \phi_Z, \mu, \zeta \rangle$ contains all information about the bialgebroid A . This is the content of the next

Theorem 1.4 *Bialgebroid structures on the ring A are in one-to-one correspondence with following categorical data.*

- i) *On the one hand a monoidal structure $\langle M_A, \square, U \rangle$ on the category of right A -modules. Denoting by Z the endomorphism ring of the monoidal unit U , every hom-set, so $A = \text{End } A_A$ too, becomes endowed with a Z - Z -bimodule structure. Thus we have the forgetful functor $\phi_Z: M_A \rightarrow {}_Z M_Z$.*
- ii) *On the other hand, a monoidal structure $\langle \phi_Z, \mu, \zeta \rangle$ on the forgetful functor such that (1.7) is exact.*

Proof The idea of the proof is to show that ϕ_Z factors through a strongly monoidal forgetful functor $\phi_R: M_A \rightarrow {}_R M_R$. Here the ring R is the additive group U together with the multiplication

$$m : U \otimes_Z U \xrightarrow{\mu_{U,U}} U \square U \xrightarrow{\sim} U \quad (1.8)$$

and unit $\zeta: Z \rightarrow U$. Every object V in M_A carries the R - R -bimodule structure defined by the left and right actions

$$\lambda_V : U \otimes_Z V \xrightarrow{\mu_{U,V}} U \square V \xrightarrow{\sim} V \quad (1.9)$$

$$\rho_V : V \otimes_Z U \xrightarrow{\mu_{V,U}} V \square U \xrightarrow{\sim} V \quad (1.10)$$

Strong monoidality of the functor $M_A \rightarrow {}_R M_R$ follows from exactness of (1.7). \square

1.2 Bialgebroids over separable base. In contrast to $A \otimes_R A$ the bimodule tensor product $A \otimes_Z A$ is a ring. This offers the tempting possibility to use, instead of γ , a comultiplication of the $A \rightarrow A \otimes_Z A$ type which could then be multiplicative in the usual sense. For this purpose we need an embedding $A \otimes_R A \subset A \otimes_Z A$. This is possible if R is a separable algebra over Z [7]. In the next definition we use [1] to formulate a Frobenius algebra structure as a special coalgebra structure on the Z -algebra R . Usually an algebra being Frobenius is a *property*, but what we need here is a *structure*, i.e., a choice of a functional ψ possessing a pair of dual bases, reformulated as a comultiplication δ .

Definition 1.5 A bialgebroid over a separable base consists of a bialgebroid $\langle A, R, t, s, \gamma, \pi \rangle$ and of a separability structure $\langle R, Z, \delta, \psi \rangle$. The latter means that the Z -algebra R has also a Z -coalgebra structure with $\delta: R \rightarrow R \otimes_Z R$ and $\psi: R \rightarrow Z$ which is compatible with the Z -algebra structure in the sense that δ is an R - R -bimodule map

$$(R \otimes_Z m) \circ (\delta \otimes_Z R) = \delta \circ m = (m \otimes_Z R) \circ (R \otimes_Z \delta) \quad (1.11)$$

(this means a Frobenius algebra structure) and moreover

$$m \circ \delta = \text{id}_R \quad (1.12)$$

(separability) where $m: R \otimes_Z R \rightarrow R$ is multiplication on R .

Notice that the data $\langle R_Z, m, \zeta, \delta, \psi \rangle$ is the same as of a bialgebra in the category of Z -modules, however, the compatibility condition between the algebra and coalgebra structures is different. The compatibility (1.11) does not need any symmetry or braiding³ in \mathbf{M}_Z .

Once having a separability structure on $R \supset Z$ we can introduce a natural transformation $\delta_{V,W}: V \otimes_R W \rightarrow V \otimes_Z W$ for R - R -bimodules V and W that splits the canonical epimorphism $V \otimes_Z W \rightarrow V \otimes_R W$. Namely

$$\delta_{V,W}(v \otimes_R w) := \sum_i v \cdot e_i \otimes_Z f_i \cdot w \quad (1.13)$$

where $\sum_i e_i \otimes_Z f_i = \delta(1_R)$. With this we can define the new comultiplication and counit on the bialgebroid A . They are the Z - Z -bimodule maps

$$\Delta := \delta_{A,A} \circ \gamma : A \rightarrow A \otimes_Z A \quad (1.14)$$

$$\varepsilon := \psi \circ \pi : A \rightarrow Z. \quad (1.15)$$

These maps no longer preserve the unit, e.g. $\Delta(1) = \sum_i s(e_i) \otimes_Z t(f_i)$, but Δ is multiplicative. The next Proposition shows how the whole bialgebroid structure of A can be reformulated in terms of Δ and ε forgetting about R altogether.

Proposition 1.6 *A bialgebroid over separable base is equivalent to the data $\langle A, Z, t, s, \Delta, \varepsilon \rangle$ where A is a ring, Z is a commutative ring, $s, t: Z \rightarrow \text{Center } A$ are unital ring homomorphisms making A into a Z - Z -bimodule, and $\langle {}_Z A_Z, \Delta, \varepsilon \rangle$ is a comonoid in the category ${}_Z \mathbf{M}_Z$. These data are subject to the axioms*

$$\Delta(ab) = \Delta(a)\Delta(b) \quad a, b \in A \quad (1.16)$$

$$(\Delta(1) \otimes_Z 1)(1 \otimes_Z \Delta(1)) = \Delta^2(1) = (1 \otimes_Z \Delta(1))(\Delta(1) \otimes_Z 1) \quad (1.17)$$

$$\varepsilon(ab_{(1)})\varepsilon(b_{(2)}c) = \varepsilon(abc) = \varepsilon(ab_{(2)})\varepsilon(b_{(1)}c) \quad a, b, c \in A, \quad (1.18)$$

where Δ^2 stands for $(\Delta \otimes_Z \text{id}) \circ \Delta \equiv (\text{id} \otimes_Z \text{id}) \circ \Delta$. Moreover, the ring Z is maximal in the sense that

$$t(Z) = A^L \cap \text{Center } A, \quad s(Z) = A^R \cap \text{Center } A \quad (1.19)$$

where the Z -subalgebras A^L of ${}_Z A$ and A^R of A_Z are defined by

$$A^L = \{a \in A \mid \Delta(a) = (a \otimes_Z 1)\Delta(1) = \Delta(1)(a \otimes_Z 1)\} \quad (1.20)$$

$$A^R = \{a \in A \mid \Delta(a) = (1 \otimes_Z a)\Delta(1) = \Delta(1)(1 \otimes_Z a)\} \quad (1.21)$$

The forgetful functor $\phi_Z: \mathbf{M}_A \rightarrow {}_Z \mathbf{M}_Z$ in the separable case is not only exactly monoidal in the sense of (1.7) but is also split in the sense of

Definition 1.7 The data $\langle F, \mu, \zeta, \delta, \psi \rangle$ is called a split monoidal functor⁴ if $\langle F, \mu, \zeta \rangle$ is a monoidal functor from a monoidal category $\langle \mathcal{C}, \square, u \rangle$ to another $\langle \mathcal{D}, \otimes, Z \rangle$, $\delta_{a,b}: F(a \square b) \rightarrow F(a) \otimes F(b)$ is a natural transformation and ψ is an arrow $F(u) \rightarrow Z$ such that

1. δ splits μ , i.e., for all objects a, b of \mathcal{C}

$$\mu_{a,b} \circ \delta_{a,b} = F(a \square b) \quad (1.22)$$

2. δ is coassociative, i.e., for all objects a, b, c of \mathcal{C}

$$(F(a) \otimes \delta_{b,c}) \circ \delta_{a,b \square c} = (\delta_{a,b} \otimes F(c)) \circ \delta_{a \square b, c} \quad (1.23)$$

³Notice that Frobenius/separability structures could have been defined in ${}_Z \mathbf{M}_Z$ which does not have any braiding. In fact this is what we do but $R \in {}_Z \mathbf{M}_Z$ is a diagonal bimodule.

⁴Another possible name: "bimonoidal functor".

3. ψ is the counit for δ , i.e.,

$$(\psi \otimes F(c)) \circ \delta_{u,c} = c = (F(c) \otimes \psi) \circ \delta_{c,u} \quad (1.24)$$

4. δ is compatible with μ in the sense of the equations

$$(\mu_{a,b} \otimes F(c)) \circ (F(a) \otimes \delta_{b,c}) = \delta_{a \square b, c} \circ \mu_{a, b \square c} \quad (1.25)$$

$$(F(a) \otimes \mu_{b,c}) \circ (\delta_{a,b} \otimes F(c)) = \delta_{a, b \square c} \circ \mu_{a \square b, c} \quad (1.26)$$

Notice that equations (1.23, 1.25, 1.26) are just variations of the associativity condition on μ in which certain μ arrows were replaced with oppositely oriented δ 's. These equations have interesting similarity with the axioms of a Frobenius structure (1.11) while the splitting property (1.22) corresponds to the separability axiom (1.12).

Split monoidal functors are just the functors arising as forgetful functors $\mathbf{M}_A \rightarrow {}_Z\mathbf{M}_Z$ for a bialgebroid over separable base. We give here the precise statement for representable functors.

Theorem 1.8 *Let $\langle \mathcal{C}, \square, u \rangle$ be a monoidal category with finite progenerator g . With the notation $Z = \text{End } u$ let $F: \mathcal{C} \rightarrow {}_Z\mathbf{M}_Z$ be the hom-functor $F = \text{Hom}(g, -)$. Then split monoidal structures $\langle F, \mu, \zeta, \delta, \psi \rangle$ for F are in bijective correspondence with bialgebroid structures $\langle {}_R A_R, \gamma, \pi \rangle$ on the ring $A = \text{End } g$ that have separable base. Moreover, any such split monoidal F factorizes, as a monoidal functor, through an equivalence $\mathcal{C} \cong \mathbf{M}_A^{f_{gp}}$ of monoidal categories to the category of finitely generated projective right A -modules.*

Proof Since g is a finite progenerator, $\mathcal{C} \cong \mathbf{M}_A^{f_{gp}}$. Monoidality of F determines the comonoid $\langle g, \gamma, \pi \rangle$ where

$$\gamma: g \rightarrow g \square g, \quad \gamma := \mu_{g,g}(g \otimes_Z g) \quad (1.27)$$

$$\pi: g \rightarrow u, \quad \pi := \zeta(u). \quad (1.28)$$

Using also the split monoidality structure we can define

$$\Delta: A \rightarrow A \otimes_Z A, \quad \Delta(a) := \delta_{g,g}(\gamma \circ a) \quad (1.29)$$

$$\varepsilon: A \rightarrow Z, \quad \varepsilon(a) := \psi(\pi \circ a) \quad (1.30)$$

which form a comonoid $\langle A, \Delta, \varepsilon \rangle$ in ${}_Z\mathbf{M}_Z$ and can be verified to obey the properties (1.16, 1.17, 1.18). The maximality property (1.19) holds automatically by the very definition of Z as $\text{End } u$. Thus A is a bialgebroid with separable base $R = \text{Hom}(g, u)$ endowed with multiplication as in (1.8). \square

Antipodes can be introduced on bialgebroids by postulating the existence of left and right dual objects in the category \mathbf{M}_A . This will not be discussed here.

1.3 Weak Hopf algebras. The (1.16, 1.17, 1.18) axioms are already very close to the weak bialgebra axioms of [3]. In fact a weak bialgebra (WBA) arises from a bialgebroid over separable base by a further finiteness condition: The commutative ring Z should be a separable algebra over a field K . Then the structure maps can be formulated in the symmetric monoidal category of K -vector spaces. Denoting the tensor product over K by \otimes there is a comultiplication $\Delta: A \rightarrow A \otimes A$ and counit $\varepsilon: A \rightarrow K$ satisfying exactly the axioms (1.16, 1.17, 1.18) except that Z is replaced everywhere with K . Then Z , more precisely two copies of it, $s(Z) = Z^R$ and $t(Z) = Z^L$, can be reconstructed as $Z^c = A^c \cap \text{Center } A$, $c = L, R$, respectively. So the (1.19) maximality condition is not needed, although it might be reasonable

to demand that the ground field be intrinsically defined by the bialgebroid structure. This could be achieved by adding the axiom that the hypercenter $Z^L \cap Z^R$ of A is a field K . (This kind of weak bialgebras (and weak Hopf algebras) are called indecomposable.)

A weak Hopf algebra (WHA) over K is a WBA A over K such that there exists a linear map $S: A \rightarrow A$, called the antipode, such that

$$a_{(1)}S(a_{(2)}) = \pi^L(a) \quad (1.31)$$

$$S(a_{(1)})a_{(2)} = \pi^R(a) \quad (1.32)$$

$$S(a_{(1)})a_{(2)}S(a_{(3)}) = S(a) \quad (1.33)$$

for all $a \in A$ where π^L, π^R are analogues of π in (1.2) and are defined by $\pi^L(a) = \varepsilon(1_{(1)}a)1_{(2)}$, $\pi^R(a) = 1_{(1)}\varepsilon(a1_{(2)})$. The antipode, if exists, is unique. It is antimultiplicative, anticomultiplicative and maps A^L onto A^R bijectively. In the sequel we shall also assume that A is finite dimensional over K . In this case S is invertible.

The dual space $\hat{A} = \text{Hom}_K(A, K)$ of a WHA endowed with multiplication and comultiplication obtained by transposing the comultiplication and multiplication of A , respectively, is again a WHA over K . Moreover there is a natural identification of their left, right subalgebras: $A^L \xrightarrow{\sim} \hat{A}^R$, $l \mapsto l \mapsto \hat{1}$ and $A^R \xrightarrow{\sim} \hat{A}^L$, $r \mapsto \hat{1} \leftarrow r$ given by the Sweedler arrows: For $a \in A$ and $\varphi \in \hat{A}$ one writes $a \mapsto \varphi := \varphi_{(1)}\langle \varphi_{(2)}, a \rangle$ and $\varphi \leftarrow a := \langle \varphi_{(1)}, a \rangle \varphi_{(2)}$.

Definition 1.9 A left (right) integral in a weak Hopf algebra A is an element $\iota_L \in A$ ($\iota_R \in A$) satisfying

$$x \iota_L = \pi^L(x) \iota_L \quad (\iota_R x = \iota_R \pi^R(x)) \quad (1.34)$$

for all $x \in A$. ι_L is called normalized if $\pi^L(\iota_L) = 1$ (ι_R is called normalized if $\pi^R(\iota_R) = 1$). A left or right integral in A is called non-degenerate if it defines a non-degenerate functional on \hat{A} .

Existence of non-zero integrals follows from a theorem on weak Hopf modules (Thm 3.9 of [3]). The existence of non-degenerate or normalized integrals in a WHA are related to the K -algebra A to be Frobenius or semisimple, respectively [3]. For example the next result provides a weak Hopf version of Maschke's Theorem.

Theorem 1.10 (see [3] Thm 3.13) *The following conditions on a WHA A over K are equivalent:*

- i) A is semisimple.
- ii) There exists a normalized left integral $\iota_L \in A$.
- iii) A is a separable K -algebra.

Definition 1.11 An element h of a WHA A is called a Haar integral in A if h is a normalized 2-sided integral, i.e., h is a left integral, a right integral, and $\pi^L(h) = \pi^R(h) = 1$.

Theorem 1.12 (see [3] Thm. 3.27) *Let A be a WHA over an algebraically closed field K . Then a necessary and sufficient condition for the existence of Haar integral $h \in A$ is that A is semisimple and there exists an invertible element $g \in A$ such that $gxg^{-1} = S^2(x)$ for $x \in A$ and $\text{tr}_r(g^{-1}) \neq 0$ in all irreducible representations r of A .*

If exists, the Haar integral is unique and is an idempotent.

2 C^* -weak Hopf algebras

A C^* -weak Hopf algebra is a WHA A over \mathbb{C} which is a finite dimensional C^* -algebra and the comultiplication Δ is a $*$ -algebra map. By uniqueness of the antipode it follows that $S(S(a)^*)^* = a$ for all $a \in A$. If also $S^2 = \text{id}$, the C^* -WHA is called a weak Kac algebra [19]. The counit $\varepsilon: A \rightarrow \mathbb{C}$ is always a positive linear functional and the associated GNS representation is the monoidal unit U of the representation category \mathbf{M}_A . If U is irreducible, or equivalently, if $Z^L = \mathbb{C}1$, i.e., the inclusion $A^L \subset A$ is connected, then A is called pure [2] or connected [19].

2.1 The Haar measure.

Theorem 2.1 *In a C^* -WHA A there exists a Haar integral. It is a selfadjoint S -invariant idempotent, $h = h^* = h^2 = S(h) \in A$, such that*

$$(\varphi, \psi) := \langle \varphi^* \psi, h \rangle, \quad \varphi, \psi \in \hat{A}, \quad (2.1)$$

is a scalar product making \hat{A} a Hilbert space and making the left regular module ${}_A \hat{A}$ a faithful $$ -representation of the $*$ -WHA \hat{A} . Thus \hat{A} is a C^* -WHA, too.*

Thus also \hat{A} has a Haar integral $\hat{h} \in \hat{A}$. This provides the faithful conditional expectations

$$E^L: A \rightarrow A^L, \quad E^L(x) = \hat{h} \rightharpoonup x \quad (2.2)$$

$$E^R: A \rightarrow A^R, \quad E^R(x) = x \leftharpoonup \hat{h} \quad (2.3)$$

It can be shown that $\hat{h} \rightharpoonup h \in A^L$ and $h \leftharpoonup \hat{h} \in A^R$ are positive and invertible. The so called canonical grouplike element is defined by

$$g := g_L g_R^{-1}, \quad (2.4)$$

$$g_L := (\hat{h} \rightharpoonup h)^{1/2}, \quad g_R := (h \leftharpoonup \hat{h})^{1/2} \quad (2.5)$$

and can be characterized as the unique $g \in A$ such that

- i) $g \geq 0$ and invertible,
- ii) $g x g^{-1} = S^2(x)$ for all $x \in A$,
- iii) $\text{tr}_r(g^{-1}) = \text{tr}_r g$ in all irreducible representations r .

In general the Haar functional $\langle \hat{h}, - \rangle: A \rightarrow \mathbb{C}$ is not a trace but instead

$$\langle \hat{h}, ab \rangle = \langle \hat{h}, b g_L g_R a (g_L g_R)^{-1} \rangle \quad a, b \in A. \quad (2.6)$$

It is a trace iff $S^2 = \text{id}$, i.e., iff A is a weak Kac algebra.

2.2 Dimensions. The category $\mathbf{rep} A$ of finite dimensional $*$ -representations of a C^* -WHA A is a monoidal category with monoidal structure inherited from the forgetful functor to the category of Hilbert A^L - A^L -bimodules, a $*$ -functor analogue of the ϕ_R of Section 1. Since the usual convention in $*$ -representations is left action, the functor is constructed by considering A to be a bimodule via $l_1 \cdot a \cdot l_2 := l_1 S^{-1}(l_2) a$, $l_1, l_2 \in A^L$, $a \in A$. Then monoidal product of two representations $D_i: A \rightarrow \mathcal{B}(H_i)$, $i = 1, 2$, is defined on the A^L - A^L -Hilbert bimodule tensor product $H_1 \otimes_{A^L} H_2$ endowed with the left action via the comultiplication, $D_1 \square D_2 := (D_1 \otimes D_2) \circ \Delta$, which is well defined due to the identities (2.31a-b [3]). The monoidal unit of $\mathbf{rep} A$ is the GNS representation D_ε associated to the counit $\varepsilon: A \rightarrow \mathbb{C}$. D_ε is irreducible iff A is pure.

All objects D of $\mathbf{rep} A$ have conjugates \bar{D} , i.e., two-sided duals, defined by help of the antipode [4]. If the WHA is pure then all conditions are fulfilled to apply the theory of dimensions of [15]. Even if A is not pure one can find analogues of

the *standard* conjugacy intertwiners $R_D: D_\varepsilon \rightarrow \bar{D} \square D$, $\bar{R}: D_\varepsilon \rightarrow D \square \bar{D}$. If D is irreducible then $R_D^* \circ R_D$ and $\bar{R}_D^* \circ \bar{R}_D$ are selfintertwiners of D_ε proportional to one minimal projection in $\text{End } D_\varepsilon = D_\varepsilon(Z^L)$ (cf. Lemma 1.3). But not to the same projection, in general. Standard normalization means choosing R_D , \bar{R}_D for all objects so that it respects direct sums, like in [15], and for irreducible objects D

$$R_D^* \circ R_D = d_D D_\varepsilon(z_\mu^L), \quad \bar{R}_D^* \circ \bar{R}_D = d_D D_\varepsilon(z_\nu^L) \quad (2.7)$$

with the same positive (in fact ≥ 1) number d_D in both equations, but with possibly different minimal projections $z_\mu^L, z_\nu^L \in Z^L$. All these data on the right hand sides depend only on the equivalence class q to which D belongs. The number $d_q = d_D$ is called the dimension of the sector q , while $q^L = \nu$ and $q^R = \mu$ are called the left and right vacuum of q , respectively. The dimensions of irreducibles can be written as

$$d_q = k(q^L)^{-1/2} k(q^R)^{-1/2} \text{tr}_q g, \quad k(\mu) := \varepsilon(z_\mu^L). \quad (2.8)$$

For pure WHA's there is only one vacuum sector. This is the case when $D \mapsto d_D$ is an additive and multiplicative dimension function. For general C^* -WHA's one forms the matrix $\mathbf{d}_q = d_q e_{\nu\mu}$ (a number times a matrix unit) for all sectors q , the rows and columns of which are labelled by the set of vacua, i.e., by the irreducibles contained in D_ε . For an arbitrary representation D one defines the matrix $\mathbf{d}_D := \sum_q N_q(D) \mathbf{d}_q$, where $N_q(D)$ is the multiplicity of q in D . The so defined dimension matrix will then be both additive and multiplicative. Conjugating the representation its dimension matrix goes to its transposed matrix.

Particularly interesting is the dimension matrix \mathbf{d}_A of the left regular representation. It turns out to be similar to the matrix $\mathbf{d}_{\hat{A}}$, which is computed, of course, in another category, in $\text{rep } \hat{A}$. But there exists a matrix \mathbf{d}^L with non-negative coefficients, and its transposed matrix \mathbf{d}^R , such that $\mathbf{d}_A = \mathbf{d}^L \mathbf{d}^R$ and $\mathbf{d}_{\hat{A}} = \mathbf{d}^R \mathbf{d}^L$. These new matrices can be interpreted as the dimension matrices of A^L and A^R , respectively [4].

In the next theorem we assume that A is an indecomposable C^* -WHA, i.e., $Z^L \cap Z^R = \mathbb{C}1$.

Theorem 2.2 (see [4]) *The basic constructions for the inclusions $A^L \subset A$ and $\hat{A} \supset \hat{A}^R$ coincide and equal to the smash product C^* -algebra $A \# \hat{A}$.*

There exists a unique normalized trace τ , called the Markov trace, on the smash product such that for all vacuum ν of A and all vacuum $\hat{\nu}$ of \hat{A} the restrictions

$$\tau \upharpoonright z_\nu^L A, \quad \tau \upharpoonright z_\nu^R A, \quad \tau \upharpoonright \hat{z}_{\hat{\nu}}^L \hat{A}, \quad \tau \upharpoonright \hat{z}_{\hat{\nu}}^R \hat{A}$$

are the Markov traces of the connected inclusions

$$z_\nu^L A^L \subset z_\nu^L A, \quad z_\nu^R A^R \subset z_\nu^R A, \quad \hat{z}_{\hat{\nu}}^L \hat{A}^L \subset \hat{z}_{\hat{\nu}}^L \hat{A}, \quad \hat{z}_{\hat{\nu}}^R \hat{A}^R \subset \hat{z}_{\hat{\nu}}^R \hat{A},$$

respectively. The corresponding trace preserving conditional expectations all have the same index δ . This index coincides with the common Perron-Frobenius eigenvalue of the regular dimension matrices \mathbf{d}_A and $\mathbf{d}_{\hat{A}}$.

If $A \cong \oplus_q M_{n_q}(\mathbb{C})$ is pure one gets the number $\delta = \mathbf{d}_A = \sum_q n_q d_q$.

The Markov conditional expectations $A \rightarrow A^L$, $A \rightarrow A^R$ are different from the Haar conditional expectations (2.2), unless A is a weak Kac algebra. In this latter case δ is an integer. The Haar conditional expectations E^L and E^R also have a common scalar index I , but $I \geq \delta$, in general.

Example 2.3 In [2] we gave an example of a C^* -WHA structure on the matrix algebra $A = M_2 \oplus M_3$. The two sectors obey the fusion rules $3 \times 3 = 2 + 3$, with 2 being the unit of the fusion ring. $A^L \cong M_1 \oplus M_1$ and $A^L \cap \text{Center } A = A^L \cap A^R = \mathbb{C}1$, so both A and \hat{A} are connected (i.e., A is biconnected). What is more $A \cong \hat{A}$. The dimensions of the sectors are $d_2 = 1$, $d_3 = (1 + \sqrt{5})/2$. The Haar index is $I = 4 + 2d_3 = 5 + \sqrt{5} = 7.24$ and the Markov index is $\delta = 2 + 3d_3 = 6.85$.

The above example is the first of a series of WHA's with the underlying algebra being a Temperley-Lieb algebra [21, 22].

Finally we mention that there exists a description of weak C^* -Hopf algebras in terms of finite dimensional multiplicative partial isometries [5, 27].

3 Finite index depth 2 inclusions

The most important (sofar the only) application of C^* -WHA's is the characterization of finite index, depth 2 inclusion of von Neumann algebras.

3.1 Weak Hopf actions. A left action of a C^* -WHA A on the unital C^* -algebra M is an algebra map $\alpha: A \rightarrow \text{End}_{\mathbb{C}} M$ which respects the $*$ -algebra structure,

$$\alpha_a(mm') = \alpha_{a_{(1)}}(m)\alpha_{a_{(2)}}(m'), \quad a \in A, m, m' \in M \quad (3.1)$$

$$\alpha_a(m)^* = \alpha_{S(a)^*}(m^*) \quad (3.2)$$

and leaves the identity "invariant" in the sense of the relation

$$\alpha_a(1_M) = \alpha_{\pi^L(a)}(1_M). \quad (3.3)$$

One also requires that $m \mapsto \alpha_a(m)$ is continuous for all $a \in A$. The invariants of a left action are the elements $n \in M$ which transform like the identity in (3.3). The invariants form a C^* -subalgebra M^A which can be expressed as the result of the application of the Haar integral, $M^A = \alpha_h(M)$.

Since the trivial representation of a WHA is not one-dimensional, together with 1_M one should consider on equal footing all operators $\alpha_l(1_M)$, $l \in A^L$. These operators form a $*$ -subalgebra M^R and $l \mapsto \alpha_l(1_M)$ is a $*$ -algebra epimorphism. (For faithful actions it is an isomorphism.) One considers M as a right A^L -module by setting $m \cdot l = m\alpha_l(1_M)$.

The crossed product C^* -algebra $M \rtimes A$ can be defined as the universal C^* -algebra of the $*$ -algebra defined on the bimodule tensor product $M \otimes_{A^L} A$ by the relations

$$(m \rtimes a)(n \rtimes b) = m\alpha_{a_{(1)}}(n) \rtimes a_{(2)}b \quad (3.4)$$

$$(m \rtimes a)^* = \alpha_{a_{(1)}^*}(m^*) \rtimes a_{(2)}^* \quad (3.5)$$

The subalgebras $1_M \rtimes A$ and $M \rtimes 1$ will be identified with A and M , respectively. One is interested in situations when the triple $M^A \subset M \subset M \rtimes A$ is a basic construction. For that we have to select a class of "nice" actions.

Definition 3.1 ([24]) A regular action of A on M is an action α such that

- i) $M^R = A^L$,
- ii) $M' \cap (M \rtimes A) = A^R$, and
- iii) $\alpha_h: M \rightarrow M^A$ is a conditional expectation of finite index [28].

Regular actions are outer in the sense of the relative commutant $M' \cap (M \rtimes A)$ being as small as possible. (A^R always commutes with M in the crossed product.) For the more fundamental meaning of outerness, as opposite to (partly) inner actions, see [24].

Proposition 3.2 ([24]) *If α is a regular action of A on the C^* -algebra M then*

1. *The crossed product $M_2 := M \rtimes A$ is the basic construction for the finite index inclusion $N := M^A \subset M$.*
2. *$N' \cap M = A^L$*
3. *$M' \cap M_2 = A^R$*
4. *$N' \cap M_2 = A$*
5. *Center $N = Z^L$, Center $M = A^L \cap A^R$, Center $M_2 = Z^R$.*

Regular actions are Galois actions: Denoting by $\rho: M \rightarrow M \otimes_{A^L} \hat{A}$ the right coaction associated to the left action α , the canonical map

$$M \otimes_N M \rightarrow M \otimes_{A^L} \hat{A}, \quad (m \otimes m') \mapsto (m \otimes \hat{1})\rho(m') \quad (3.6)$$

is an isomorphism. For the proof see the Appendix of [24].

3.2 Depth 2 inclusions of II_1 factors. This subsection is based on the results obtained by D. Nikshych and L. Vainerman in [20, 21]. Let $N \subset M$ be a finite index, depth 2 inclusion of II_1 factors and let $E: M \rightarrow N$ denote the trace preserving conditional expectation. Then one constructs the Jones tower

$$N \overset{E_1}{\subset} M \overset{E_2}{\subset} M_2 \overset{E_3}{\subset} M_3 \subset \dots$$

with the finite index conditional expectations $E_n: M_n \rightarrow M_{n-1}$ implemented by Jones projections $e_n \in M'_{n-1} \cap M_{n+1}$ satisfying the Temperley-Lieb algebra with $e_n e_{n+1} e_n = e_n / \delta$, where δ is the index of E , i.e., the minimal index of $N \subset M$. The derived tower $\dots \subset N' \cap M_n \subset N' \cap M_{n+1} \subset \dots$ consists of finite dimensional C^* -algebras and is a Jones tower starting from $N' \cap M$ (depth 2 condition).

Define the algebras $A := N' \cap M_2$, $B := M'_1 \cap M_3$ and a pairing, i.e., a non-degenerate bilinear form (cf Eqn 4.134 in [4])

$$\langle a, b \rangle := \delta^{3/2} \tau(e_2 e_1 b c_L a c_R), \quad a \in A, b \in B, \quad (3.7)$$

where $c_L \in \text{Center}(N' \cap M)$, $c_R \in \text{Center}(M' \cap M_2)$ are positive invertible elements. The pairing yields the coalgebras $\langle A, \Delta_A, \varepsilon_A \rangle$ and $\langle B, \Delta_B, \varepsilon_B \rangle$ immediately. Antipodes can be introduced by

$$S_A: A \rightarrow A, \quad S_A(a) := (a_*)^* \quad \text{where } \langle a_*, b \rangle := \overline{\langle a, b^* \rangle}$$

and an analogue expression for $S_B: B \rightarrow B$. The difficult part is to show that for an appropriate (in fact unique) choice of c_L, c_R the comultiplication Δ_B is a $*$ -algebra map. If it is done all other axioms of weak Hopf algebras hold automatically. The next Theorem is a reformulation of the results of D. Nikshych and L. Vainerman and provides a generalization of the duality theorem of irreducible inclusions of factors and C^* -Hopf algebra actions [25, 14, 6].

Theorem 3.3 (cf. [20]) *Let $N \subset M \subset M_2 \subset M_3$ be the Jones tower built on a finite index, depth 2 inclusion $N \subset M$ of II_1 factors. Then there are biconnected C^* -WHA structures on the relative commutants $A = N' \cap M_2$ and $B = M' \cap M_3$ such that*

1. *A and B are the duals of each other w.r.t. the pairing (3.7)*

2. A acts regularly on M with N being the invariant subalgebra
3. M_2 is isomorphic to the crossed product $M \rtimes A$
4. the index $[M : N]$ is equal to the Markov index δ of the finite dimensional inclusion $A^L \subset A$ determined by Thm 2.2.

The WHA's become unique under the additional requirement of $S^2 \upharpoonright A^L$ be the identity.

Although the generalization to non-factors is quite plausible and has already been suggested in [24], no published results are available yet. There is, however, another approach by M. Enock and J.-M. Vallin by means of which almost arbitrary depth 2 inclusions can be described as invariant subalgebras w.r.t. actions of Hopf bimodules [9, 10].

In [21] Nikshych and Vainerman made an important step in another direction. They considered any finite depth inclusion $N \subset M$ of II_1 factors. Since there is always a depth 2 subtower $N \subset M_p \subset M_{2p} \subset \dots$ of the Jones tower over $N \subset M_1$, there is always a WHA A acting on M_p with N being its invariant subalgebra. Let its dual weak Hopf algebra $M'_p \cap M_{3p}$ be denoted by B . The question arises how the intermediate factors M_k , $2p < k < 3p$ are related to substructures of B ? The appropriate substructure is a *left coideal *-subalgebra*, i.e., a unital *-subalgebra $I \subset B$ such that $\Delta(I) \subset B \otimes I$. These coideal subalgebras form a lattice with minimal element B^L and maximal element B . The next theorem extends the earlier results of [13, 11].

Theorem 3.4 (see [21]) *Let $N \subset M_1 \subset M_2 \subset M_3 \subset \dots$ be the tower constructed from a depth 2 subfactor $N \subset M_1$ and let $B = M'_1 \cap M_3$ be the corresponding quantum groupoid. Then intermediate factors $M_2 \subset K \subset M_3$ and left coideal *-subalgebras $I \subset B$ are in one-to-one correspondence via*

$$K \mapsto I = M'_1 \cap K \subset B \quad \text{and} \quad I \mapsto K = M_2 \rtimes I \subset M_3 .$$

3.3 Abstract inclusions. The above results are very probably only special cases of a much more general duality between inclusions and quantum groupoid actions. I try to outline here the construction of bialgebroids from depth 2 arrows in a 2-category. The data what one needs for this construction to work is reminiscent to the data of an abstract Q-system proposed by Longo in [14], although we work in the non-* framework.

Let $a: M \rightarrow N$ be an arrow in an additive 2-category \mathcal{C} . Assume a has a left dual $a^L: N \rightarrow M$ with unit $\eta: N \rightarrow a \square a^L$ and counit $\varepsilon: a^L \square a \rightarrow M$. (One may think \mathcal{C} to be the 2-category of categories and a to be the forgetful functor corresponding to a ring inclusion $N \rightarrow M$. Then a^L is the induction functor.)

Then $g := a^L \square a$ has a comonoid structure in the monoidal category $\mathcal{C}(M, M)$ defined by

$$\delta := a^L \square \eta \square a \quad : g \rightarrow g \square g \tag{3.8}$$

$$\pi := \varepsilon \quad : g \rightarrow M \tag{3.9}$$

Let F denote the hom-functor $\text{Hom}(g, -)$ from $\mathcal{C}(M, M)$ to the category ${}_R \mathbf{M}_R$ of R - R -bimodules where $R := \text{End } a$ and the bimodule structure on a hom-group is defined by

$$r \cdot h := h \circ (r^L \square a), \quad h \cdot r := h \circ (a^L \square r). \tag{3.10}$$

Here $r \mapsto r^L$ denotes the action of the left dual functor mapping R to $\text{End } a^L$. Clearly, F factorizes through the category of right $A := \text{End } g$ -modules:

$$\begin{array}{ccc} \mathcal{C}(M, M) & \xrightarrow{F} & {}_R\mathbf{M}_R \\ & \searrow & \nearrow \phi \\ & \mathbf{M}_A & \end{array} \quad (3.11)$$

where the forgetful functor ϕ is meant w.r.t. the bimodule structure on A as in Definition 1.1 with source and target maps

$$t(r) := r^L \square a, \quad s(r) := a^L \square r. \quad (3.12)$$

In order to demonstrate that ϕ is monoidal we need the natural transformation

$$\mu_{b,c} : F(b) \otimes_R F(c) \rightarrow F(b \square c) \quad (3.13)$$

$$\mu_{b,c}(x \otimes_R y) = (x \square y) \circ \delta \quad (3.14)$$

which is well-defined due to $(r \square a^L) \circ \eta = (a \square r^L) \circ \eta$, and it is an R - R -bimodule map. Together with the arrow

$$\nu : R \rightarrow F(M), \quad r \mapsto \varepsilon \cdot r = r \cdot \varepsilon = \varepsilon \circ (a^L \square r) \quad (3.15)$$

μ satisfies associativity and unit constraints establishing the monoidal functor $\langle \phi, \mu, \nu \rangle$.

ν is in fact an isomorphism. In order for μ to be also an isomorphism we have to make a further assumption on the arrow a and restrict F to an appropriate subcategory of $\mathcal{C}(M, M)$. Assume that a is of *depth 2*, i.e., $a \square a^L \square a$ is a direct summand of a finite direct sum of a 's. Let \mathcal{C}_g be the full subcategory of $\mathcal{C}(M, M)$ the objects b of which are direct summands of a finite multiple of g 's. Then \mathcal{C}_g contains the tensor powers $g^{\square n}$, is a subcategory closed under the monoidal product, and has subobjects.

We need yet a further assumption, namely that M belongs to \mathcal{C}_g . This is equivalent to the assumption that M is contained in g as a direct summand.

Theorem 3.5 *Let \mathcal{C} be an additive 2-category⁵ closed under direct sums and subobjects of arrows (= 1-cells) and let the arrow $a : N \rightarrow M$ possess a left dual, be of depth 2, and be such that M is contained in $g = a^L \square a$ as a direct summand. Then there is a full monoidal subcategory \mathcal{C}_g of $\mathcal{C}(M, M)$ which is equivalent, as a monoidal category, to the category $\mathbf{M}_A^{\text{f.g.p.}}$ of finitely generated projective right A -modules for a uniquely determined bialgebroid structure on $A = \text{End } g$ over the base $R = \text{End } a$. The functor $\text{Hom}(g, -) : \mathcal{C}_g \rightarrow {}_R\mathbf{M}_R$ factorizes, as a monoidal functor, through the forgetful functor $\phi : \mathbf{M}_A \rightarrow {}_R\mathbf{M}_R$.*

Proof Need to show that $\mu_{b,c}$ is an isomorphism for objects b, c of \mathcal{C}_g . Choosing a direct sum decomposition $a \square b \xrightarrow{e_i} a \xrightarrow{f_i} a \square b$ we can explicitly write down the inverse as

$$\mu_{b,c}^{-1}(t) = \sum_i (\varepsilon \square b) \circ (a^L \square f_i) \otimes_R (\varepsilon \square c) \circ (a^L \square e_i \square c) \circ (g \square t) \circ (a^L \square \eta \square a)$$

for $t \in \text{Hom}(g, b \square c)$. This proves strong monoidality of $\langle F \upharpoonright \mathcal{C}_g, \mu, \nu \rangle$. From the construction of \mathcal{C}_g it is clear that $\mathcal{C}_g \cong \mathbf{M}_A^{\text{f.g.p.}}$, as categories. Use this equivalence

⁵In fact the theorem holds for bicategories, too.

to define a monoidal structure on $M_A^{\text{f.g.p.}}$. Now apply Lemma 1.2 to conclude that A is a bialgebroid over R and the equivalence is that of monoidal categories. The monoidal factorization through ϕ holds by the very definition of the monoidal structure of $M_A^{\text{f.g.p.}}$. \square

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